The ashes of this fossil, when burnt, being boiled in water, and the water evaporated, there remained no salt behind.

I am, my Lord, &c.

Grosvenor-Street, Feb. 28, 1760.

LIV. *A new Method of computing the Sums of certain Series; by Mr.* John Landen: *Communicated by Mr.* Thomas Simpson, *F. R. S.*

Read Feb. 28, 1760.

AS the improving the analytic art, especially any branch of it that relates to the summation of series, may, by facilitating computations, conduce to the improvement of several branches of science; it is presumed, that this paper, which exhibits a new and easy method of computing the sums of a great number of infinite series, may be acceptable to the mathematical world, and deemed worthy to be inserted in the British Philosophical Transactions.

1.

Supposing x to be the sine of the circular arc z, whose radius is 1, $\frac{\dot{x}}{\sqrt{1-x^2}}$ will be $=\dot{z}$; and, consequently, $\frac{\dot{x}}{\sqrt{x^2-1}}=\frac{\dot{z}}{\sqrt{-1}}$. From whence, by taking the correct fluents, we have hyp. log. $\frac{x+\sqrt{x^2-1}}{\sqrt{-1}}=\frac{z}{\sqrt{-1}}$.

Hence,

Hence, writing a for one fourth of the periphery of the circle whose radius is 1, and taking x equal to the said radius, we find hyp. log. $\frac{1}{\sqrt{-1}} = \frac{a}{\sqrt{-1}}$; and, consequently, hyp. log. $\sqrt{-1} = \frac{-a}{\sqrt{-1}}$, and hyp. log. $-1 = \pm\frac{2a}{\sqrt{-1}}$.

2.

The hyp. log. of $\frac{1}{1-x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$, &c.

$\overset{\text{I}}{F}$ = fluent of $\frac{\dot{x}}{x}$ hyp. log. $\frac{1}{1-x}$, is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2}$, &c.

$\overset{\text{II}}{F}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{I}}{F} = x + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \frac{x^4}{4^3}$, &c.

$\overset{\text{III}}{F}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{II}}{F} = x + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \frac{x^4}{4^4}$, &c.

$\overset{\text{IV}}{F}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{III}}{F} = x + \frac{x^2}{2^5} + \frac{x^3}{3^5} + \frac{x^4}{4^5}$, &c.

&c. &c. &c.

3.

By writing, in the first equation in the preceding article, $\frac{1}{x}$ instead of x, we have

Hyp. log. $\frac{1}{1-\frac{1}{x}} = x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}$, &c.

But the hyp. log. of $\frac{1}{1-\frac{1}{x}}$ is = hyp. log. $\frac{x}{x-1}$ =

hyp.

hyp. log. $\frac{1}{1-x}$ + hyp. log. x + hyp. log. $-1 =$ $\pm 2b + X +$ hyp. log. $\frac{1}{1-x}$, b being put for $\frac{a}{\sqrt{-1}}$, and X for the hyp. log. of x.

It is evident, therefore, that

Hyp. log. $\frac{1}{1-x}$ is $= \mp 2b - X + x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3}$, &c. where, of the two signs prefixed to $2b$, the upper one takes place, when the hyp. log. of -1 is taken equal to $\frac{-2a}{\sqrt{-1}}$, likewise when x is taken equal to $\sqrt{-1}$; and the lower one takes place, when the hyp. log. of -1 is taken equal to $\frac{2a}{\sqrt{-1}}$, also when x is taken equal to $\frac{1}{\sqrt{-1}}$: wherefore, if we observe to take the value of hyp. log. of -1, as last mentioned, and x equal to $\frac{1}{\sqrt{-1}}$, instead of $\sqrt{-1}$, we need retain only the lower of the said signs.

4.

For brevity sake, we shall, in what follows, put the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. $= \overset{II}{P}$,

$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}$, &c. $= \overset{IV}{P}$,

$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6}$, &c. $= \overset{VI}{P}$,

&c. &c.

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$, &c. $= \overset{II}{Q}$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} +, \text{ \&c.} = \overset{\text{III}}{q},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4}, \text{ \&c.} = \overset{\text{IV}}{Q}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} +, \text{ \&c.} = \overset{\text{V}}{q},$$

&c. &c.

5.

Multiplying the laſt equation in art. 3. by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\text{I}}{F} = 2\overset{\text{II}}{P} + 2bX - \frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}, \text{ \&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\text{II}}{F} = 2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1} + \frac{x^{-2}}{2^3} + \frac{x^{-3}}{3^3}, \text{ \&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\text{III}}{F} = 2\overset{\text{IV}}{P} + \overset{\text{II}}{P}X^2 + \frac{bX^3}{3} - \frac{X^4}{2.3.4} - x^{-1} - \frac{x^{-2}}{2^4} - \frac{x^{-3}}{3^4}, \text{ \&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\text{IV}}{F} = 2\overset{\text{IV}}{P}X + \frac{\overset{\text{II}}{P}X^3}{3} + \frac{bX^4}{3.4} - \frac{X^5}{2.3.4.5} + x^{-1} + \frac{x^{-2}}{2^5} + \frac{x^{-3}}{3^5}, \text{ \&c.}$$

&c. &c.

6.

Now, it is obvious, that $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. the value of $\overset{\text{I}}{F}$ in art. 2. muſt be equal to $2\overset{\text{II}}{P} + 2bX$

$-\frac{X^2}{2} - x^{-1} - \frac{x^{-2}}{2^2} - \frac{x^{-3}}{3^2}$, &c. the value of F in art. 5. when both series converge.

Therefore, $\frac{x + x^{-1}}{1^2} + \frac{x^2 + x^{-2}}{2^2} + \frac{x^3 + x^{-3}}{3^2}$, &c. is then $= 2\overset{\text{II}}{P} + 2bX - \frac{X^2}{2}$.

From which equation, by taking x equal to -1, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{\text{II}}{P} + b^2 = \overset{\text{II}}{P} - a^2$; and, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= 4\overset{\text{II}}{P} + 3b^2 = 4\overset{\text{II}}{P} - 3a^2$.

Therefore $4\overset{\text{II}}{P} - 3a^2$ is $= \overset{\text{II}}{P} - a^2$:

Hence $\overset{\text{II}}{P}$ is found $= \frac{2a^2}{3}$.

Moreover $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. being $= \overset{\text{II}}{P}$, by supposition; and $-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} -$, &c. $= \overset{\text{II}}{P} - a^2$, as found above; we, by subtraction, get $\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$, &c. $(= 2\overset{\text{II}}{Q}) = a^2$, and, consequently $\overset{\text{II}}{Q} = \frac{a^2}{2}$.

SCHOLIUM.

The hyp. log. of $\frac{1}{1-x}$ being $= x + \frac{x^2}{2} + \frac{x^3}{3}$, &c. we, by writing $1 - x$ instead of x, have

Hyp. log. of $\frac{1}{x} = 1 - x + \frac{\overline{1-x}|^2}{2} + \frac{\overline{1-x}|^3}{3}$, &c.
and consequently $X = -1 - x - \frac{\overline{1-x}|^2}{2} - \frac{\overline{1-x}|^3}{3}$, &c.

Moreover the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. which vanishes when x vanishes; and the fluent of $\frac{\dot{x}}{1-x} \times X$ is $= \overline{1-x} + \frac{\overline{1-x}|^2}{2^2} + \frac{\overline{1-x}|^3}{3^2}$, &c. $- \overset{\text{II}}{P}$, being corrected so as to vanish when x vanishes.

But the fluent of $\frac{\dot{x}}{x} \times$ hyp. log. of $\frac{1}{1-x}$ + fluent of $\frac{\dot{x}}{1-x} \times X$ is $= X \times$ hyp. log. of $\frac{1}{1-x}$, which also vanishes when x vanishes.

Therefore $X \times$ hyp. log. of $\frac{1}{1-x}$ is $= x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. $+ 1 - x + \frac{\overline{1-x}|^2}{2^2} + \frac{\overline{1-x}|^3}{3^2}$, &c. $- \overset{\text{II}}{P}$.

From whence, by taking x equal to $\frac{1}{2}$, we find $-$ square of hyp. log. of $2 = 2 \times \frac{1}{1^2.2^1} + \frac{1}{2^2.2^2} + \frac{1}{3^2.2^3}$, &c. $- \overset{\text{II}}{P}$: hence, $\overset{\text{II}}{P}$ being before found $= \frac{2a^2}{3}$, it appears that, when x is $= \frac{1}{2}$, the series $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$, &c. is $= \frac{a^2}{3} - \frac{1}{2} \times \overline{\text{hyp. log. of } 2}|^2$.

7.

Furthermore, $x + \frac{x^2}{2^3} + \frac{x^3}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 2. must be equal to $2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3} + x^{-1}$

+

$+\frac{x^{-2}}{2^3}+\frac{x^{-3}}{3^3}$, &c. the value of $\overset{\text{II}}{F}$ in art. 5. when both series converge.

Therefore $\frac{x - x^{-1}}{1^3}+\frac{x^2 - x^{-2}}{2^3}+\frac{x^3 - x^{-3}}{3^3}$, &c. is then $= 2\overset{\text{II}}{P}X + bX^2 - \frac{X^3}{2.3}$.

From whence, by taking x equal to -1, we have $4b\overset{\text{II}}{P} + 4b^3 - \frac{8b^3}{2.3} = 0$; and, consequently, $\overset{\text{II}}{P} = \frac{2a^2}{3}$, as before found.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$\frac{2}{\sqrt{-1}} \times \overset{\text{III}}{q} = 2b\overset{\text{II}}{P} + b^3 - \frac{b^3}{2.3} = \frac{4a^3}{3\sqrt{-1}} - \frac{a^3}{\sqrt{-1}} + \frac{a^3}{2.3\sqrt{-1}} = \frac{a^3}{2\sqrt{-1}}.$$

Therefore $\overset{\text{III}}{q}$ is $= \frac{a^3}{4}$.

8.

From what is done above, it evidently follows, that

$$-\overset{\text{IV}}{P} \text{ is} = \frac{2b^2\overset{\text{II}}{P}}{3} + \frac{2.8b^4}{3.4.5},$$

$$-\overset{\text{VI}}{P} = \frac{2b^2\overset{\text{IV}}{P}}{3} + \frac{8b^4\overset{\text{II}}{P}}{3.4.5} + \frac{3.32b^6}{3.4.5.6.7},$$

&c. &c.

$$-\overset{\text{IV}}{Q} = b^2\overset{\text{II}}{P} + \frac{3.2b^4}{3.4},$$

$$-\overset{\text{VI}}{Q} = b^2\overset{\text{IV}}{P} + \frac{4b^4\overset{\text{II}}{P}}{3.4} + \frac{5.8b^6}{3.4.5.6},$$

&c. &c.

$$\frac{\overset{\text{V}}{q}}{\sqrt{-1}} \text{ is} = b\overset{\text{IV}}{P} + \frac{b^3 \overset{\text{II}}{P}}{2.3} + \frac{9\,b^5}{2.2.3.4.5},$$

$$\frac{\overset{\text{VII}}{q}}{\sqrt{-1}} = b\overset{\text{VI}}{P} + \frac{b^3 \overset{\text{IV}}{P}}{2.3} + \frac{b^5 \overset{\text{II}}{P}}{2.3.4.5} + \frac{13\,b^7}{2.2.3.4.5.6.7},$$

&c. &c.

From whence the values of $\overset{\text{IV}}{P}$, $\overset{\text{VI}}{P}$, &c. $\overset{\text{IV}}{Q}$, $\overset{\text{VI}}{Q}$, &c. $\overset{\text{V}}{q}$, $\overset{\text{VII}}{q}$, &c. may be easily obtained, in terms of a.

9.

Hyp. log. $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}}$ being $= x + \frac{x^3}{3} + \frac{x^5}{5}$, &c.

$$\overset{\text{I}}{G} = \text{fluent of } \frac{\dot{x}}{x} \text{ hyp. log. } \overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}} \text{ is} = x + \frac{x^3}{3^2} + \frac{x^5}{5^2}, \text{ \&c.}$$

$$\overset{\text{II}}{G} = \text{fluent of } \frac{\dot{x}}{x}\overset{\text{I}}{G} = x + \frac{x^3}{3^3} + \frac{x^5}{5^3}, \text{ \&c.}$$

$$\overset{\text{III}}{G} = \text{fluent of } \frac{\dot{x}}{x}\overset{\text{II}}{G} = x + \frac{x^3}{3^4} + \frac{x^5}{5^4}, \text{ \&c.}$$

&c. &c. &c.

10.

By writing, in the first equation in the preceding article, $\frac{1}{x}$ instead of x, we have

$$\text{Hyp. log. } \overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\Bigg|^{\frac{1}{2}} = x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}, \text{ \&c.}$$

But the hyp. log. of $\overline{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\Big|^{\frac{1}{2}}$ is $=$ hyp. log. $\overline{\frac{x+1}{x-1}}\Big|^{\frac{1}{2}}$

$=$ hyp. log. $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}} \mp$ hyp. log. $\sqrt{-1} = \pm\, b +$ hyp. log. $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}}$.

It

It is manifeſt, therefore, that

Hyp. log. $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}}$ is $= \mp b + x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5}$, &c. where, with reſpect to the two ſigns prefixed to b, the ſame obſervation may be made as in art. 3.

11.

Multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we have

$$\overset{\mathrm{I}}{G} = 2\overset{\mathrm{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}, \text{ \&c.}$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the fluents, we get

$$\overset{\mathrm{II}}{G} = 2\overset{\mathrm{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3} + \frac{x^{-5}}{5^3}, \text{ \&c.}$$

Again, multiplying the laſt equation by $\frac{\dot{x}}{x}$, and taking the correct fluents, we find

$$\overset{\mathrm{III}}{G} = 2\overset{\mathrm{IV}}{Q} + \overset{\mathrm{II}}{Q}X^2 + \frac{bX^3}{2.3} - x^{-1} - \frac{x^{-3}}{3^4} - \frac{x^{-5}}{5^4}, \text{ \&c.}$$

And, by proceeding in the ſame manner, we find

$$\overset{\mathrm{IV}}{G} = 2\overset{\mathrm{IV}}{Q}X + \frac{\overset{\mathrm{II}}{Q}X^3}{3} + \frac{bX^4}{2.3.4} + x^{-1} + \frac{x^{-3}}{3^5} + \frac{x^{-5}}{5^5}, \text{ \&c.}$$

&c. &c.

12.

Now, it is obvious, that $x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. the value of $\overset{\mathrm{I}}{G}$ in art. 9. muſt be equal to $2\overset{\mathrm{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. the value of $\overset{\mathrm{I}}{G}$ in art. 11. when both ſeries converge

There-

Therefore $\frac{x + x^{-1}}{1^2} + \frac{x^3 + x^{-3}}{3^2} + \frac{x^5 + x^{-5}}{5^2}$, &c. is then $= 2\overset{\text{II}}{Q} + bX$.

From whence, by taking x equal to $\frac{1}{\sqrt{-1}}$, we have $2\overset{\text{II}}{Q} + b^2 = 0$; and, consequently, $\overset{\text{II}}{Q} = \frac{a^2}{2}$, as in art. 6.

13.

Likewise $x + \frac{x^3}{3^3} + \frac{x^5}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 9. must be equal to $2\overset{\text{II}}{Q}X + \frac{bX^2}{2} + x^{-1} + \frac{x^{-3}}{3^3} + \frac{x^{-5}}{5^3}$, &c. the value of $\overset{\text{II}}{G}$ in art. 11. when both series converge.

Therefore $\frac{x - x^{-1}}{1^3} + \frac{x^3 - x^{-3}}{3^3} + \frac{x^5 - x^{-5}}{5^3}$, &c. is then $= 2\overset{\text{II}}{Q}X + \frac{bX^2}{2}$.

Hence, by taking $x = \frac{1}{\sqrt{-1}}$, we find $\frac{2}{\sqrt{-1}} \times \overset{\text{III}}{q} = 2b\overset{\text{II}}{Q} + \frac{b^3}{2} = \frac{a^3}{2\sqrt{-1}}$; and, consequently, $\overset{\text{III}}{q} = \frac{a^3}{4}$, as in art. 7.

14.

From what is done in the last five articles, it evidently follows, that

$$-\overset{\text{IV}}{Q} \text{ is} = \frac{b^2\overset{\text{II}}{Q}}{2} + \frac{b^4}{2.2.3},$$

$$\frac{\overset{\text{V}}{q}}{\sqrt{-1}} = b\overset{\text{IV}}{Q} + \frac{b^3\overset{\text{II}}{Q}}{2.3} + \frac{b^5}{2.2.3.4},$$

$$-\overset{\text{VI}}{Q} \text{ is} = \frac{b^2\overset{\text{IV}}{Q}}{2} + \frac{b^4\overset{\text{II}}{Q}}{2.3.4} + \frac{b^6}{2.\,.3.4.5},$$

$$\frac{\overset{\text{VII}}{q}}{\sqrt{-1}} = b\overset{\text{VI}}{Q} + \frac{b^3\overset{\text{IV}}{Q}}{2.3} + \frac{b^5\overset{\text{II}}{Q}}{2.3.4.5} + \frac{b^7}{2.2.3.4.5.6},$$

&c. &c.

From whence (as well as from the theorems in art. 8.) may the values of $\overset{\text{IV}}{Q}$, $\overset{\text{V}}{q}$, $\overset{\text{VI}}{Q}$, $\overset{\text{VII}}{q}$, &c. be readily found, in terms of a.

15.

$\overset{\text{I}}{G}$ being $= x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$, &c. by art. 9.

$\overset{\text{I}}{H}$ = fluent of $x\dot{x}\overset{\text{I}}{G}$ is $= \frac{x^3}{1^2.3} + \frac{x^5}{3^2.5} + \frac{x^7}{5^2.7}$, &c.

$\overset{\text{II}}{H}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{I}}{H} = \frac{x^3}{1^2.3^2} + \frac{x^5}{3^2.5^2} + \frac{x^7}{5^2.7^2}$, &c.

$\overset{\text{III}}{H}$ = fluent of $x\dot{x}\overset{\text{II}}{H} = \frac{x^5}{1^2.3^2.5} + \frac{x^7}{3^2.5^2.7} + \frac{x^9}{5^2.7^2.9}$, &c.

$\overset{\text{IV}}{H}$ = fluent of $\frac{\dot{x}}{x}\overset{\text{III}}{H} = \frac{x^5}{1^2.3^2.5^2} + \frac{x^7}{3^2.5^2.7^2} + \frac{x^9}{5^2.7^2.9^2}$, &c.

&c. &c. &c.

16.

Moreover, $\overset{\text{I}}{G}$ being $= 2\overset{\text{II}}{Q} + bX - x^{-1} - \frac{x^{-3}}{3^2} - \frac{x^{-5}}{5^2}$, &c. by art. 11. we, by multiplying by $x\dot{x}$, and taking the correct fluents, get $\overset{\text{I}}{H} = x^2\overset{\text{II}}{Q} - \overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\text{II}}{S} + \frac{x^{-1}}{1.3^2} + \frac{x^{-3}}{3.5^2} + \frac{x^{-5}}{5.7^2}$, &c. $\overset{\text{II}}{S}$ being put for the series $\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2}$, &c.

Now,

Now, it is obvious, that this value of $\overset{\text{I}}{H}$ must be equal to the value of $\overset{\text{I}}{H}$ in the preceding article, when both series converge.

Therefore $\frac{3x^3 - x^{-1}}{1^2.3^2} + \frac{5x^5 - 3x^{-3}}{3^2.5^2} + \frac{7x^7 - 5x^{-5}}{5^2.7^2}$, &c. is then $= x^2\overset{\text{II}}{Q} - \overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bx^2}{4} + \frac{b}{4} - x + 1 + 2\overset{\text{II}}{S}$.

Hence, by taking x equal to -1, we find $-2\overset{\text{II}}{S} = b^2 + 2 + 2\overset{\text{II}}{S}$; and, consequently, $\overset{\text{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$.

And, by taking x equal to $\frac{1}{\sqrt{-1}}$, we find

$$-\frac{2}{\sqrt{-1}} \times \overline{\frac{1}{1^2.3^2} - \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} -, \&c.} = -2\overset{\text{II}}{Q} - \frac{b^2}{2} + \frac{b}{2} - \frac{1}{\sqrt{-1}} + 1 + 2\overset{\text{II}}{S} = \frac{a}{2\sqrt{-1}} - \frac{1}{\sqrt{-1}};$$

and, consequently, $\frac{1}{1^2.3^2} - \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} -$, &c. $= \frac{1}{2} - \frac{a}{4}$.

17.

Seeing that $\overset{\text{II}}{Q}$ is $= \frac{a^2}{2}$, and $\overset{\text{II}}{S} = \frac{a^2}{4} - \frac{1}{2}$, it follows, from the last article, that

$$\overset{\text{I}}{H} \text{ is } = x^2\overset{\text{II}}{Q} + \frac{bx^2X}{2} - \frac{bX^2}{4} + \frac{b}{4} - x + \frac{x^{-1}}{1.3^2} + \frac{x^{-3}}{3.5^2} + \frac{x^{-5}}{5.7^2}, \&c.$$

From whence, by multiplying by $\frac{\dot{x}}{x}$, and taking the correct fluent, we get

$\overset{\text{II}}{H} = \frac{x^2 \overset{\text{II}}{Q}}{2} + \frac{b x^2 X}{4} - \frac{b x^2}{4} + \frac{b}{4} + \frac{b X}{4} - x + \frac{a^2}{4}$
$- \frac{x^{-1}}{1^2.3^2} - \frac{x^{-3}}{3^2.5^2} - \frac{x^{-5}}{5^2.7^2}$, &c.

And hence, by multiplying by $x \dot{x}$, and taking the correct fluents, we have

$\overset{\text{III}}{H} = \frac{x^4 \overset{\text{II}}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64}$
$- \frac{x^3}{3} - \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} - \frac{3 a^2}{16} + 4 \overset{\text{III}}{S} + \frac{x^{-1}}{1 \cdot 3^2 \cdot 5^2}$
$+ \frac{x^{-3}}{3 \cdot 5^2 \cdot 7^2} + \frac{x^{-5}}{5 \cdot 7^2 \cdot 9^2}$, &c. $\overset{\text{III}}{S}$ being put for the series
$\frac{1}{1^2.3^2.5^2} + \frac{1}{3^2.5^2.7^2} + \frac{1}{5^2.7^2.9^2}$, &c.

Now, this value of $\overset{\text{III}}{H}$ being equal to the value of $\overset{\text{III}}{H}$ in art. 15. when both series converge, it follows, that $\frac{5 x^5 - x^{-1}}{1^2.3^2.5^2} + \frac{7 x^7 - 3 x^{-3}}{3^2.5^2.7^2} + \frac{9 x^9 - 5 x^{-5}}{5^2.7^2.9^2}$, &c. is then
$= \frac{x^4 \overset{\text{II}}{Q}}{8} + \frac{b x^4 X}{16} - \frac{5 b x^4}{64} + \frac{b x^2}{16} + \frac{b x^2 X}{8} + \frac{b}{64} - \frac{x^3}{3}$
$- \frac{x}{9} + \frac{4}{9} + \frac{a^2 x^2}{8} + \frac{3 a^2}{16} - 4 \overset{\text{III}}{S}$.

Hence, by taking x equal to -1, we find $-4 \overset{\text{III}}{S}$ $= \frac{3 b^2}{8} + \frac{8}{9} + 4 \overset{\text{III}}{S}$; and, consequently, $\overset{\text{III}}{S} = \frac{3 a^2}{64} - \frac{1}{9}$.

Many other instances of the use of this method might be given; but these may suffice to enable the intelligent reader to pursue the speculation farther, at his pleasure.